Database Theory
Winter Term 2013/14
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3. Unit: Model Theory and Reasoning

Discussion by 11./18.12.2013

Exercise 1 (FOL: Entailment + Tableau) For the following pairs F and G of formulas, check whether one implies the other (if not, give a counterexample), and whether they are equivalent.

Solve the questions first by thinking, and then prove by using the tableau calculus.

a) $F = (\forall x : p(x)) \vee (\forall x : q(x))$, $G = \forall v : (p(v) \vee q(v))$.

b) $F = \forall x : ((\exists y : p(y)) \rightarrow q(x))$, $G = \forall v, w : (p(v) \rightarrow q(w))$.

c) $F = \forall x \exists y : p(x, y)$, $G = \exists w \forall v : p(v, w)$.

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a) For an intuitive idea replace $p(d)$ by "d is male" und for $q(d)$ "d is female". If all d are male or all d are female, every individual d is also male or female. In reverse direction, d_1 can be male and d_2 can be female.
$F \models G$ holds:
If all elements of the domain are in p or all are in q, every individual one is also in p or in q.
$G \models F$ does not hold:
Let $\mathcal{I} = (I, \mathcal{D})$ with $\mathcal{D} = \{1, 2\}$, $I(p) = \{(1)\}$, $I(q) = \{(2)\}$.
$\mathcal{I} \models G$, but not $\mathcal{I} \models F$.
Thus, F and G are not equivalent.
As tableau $F \models G$:
Start with $F \wedge \neg G$.

$$((\forall x : p(x)) \vee (\forall x : q(x))) \wedge \neg \forall v : (p(v) \vee q(v)) \quad (1)$$
$$(\forall x : p(x)) \vee (\forall x : q(x)) \quad (2)$$
$$\neg \forall v : (p(v) \vee q(v)) \quad (3)$$
resolve first (3), since (2) would branch:
$$\exists v : \neg(p(v) \vee q(v)) \quad (4)$$
Skolem constant, since no free vars:
$$\neg(p(a) \vee q(a)) \quad (5)$$
$$\neg p(a) \quad (6)$$
$$\neg q(a) \quad (7)$$
now (2):

/	\
$\forall x : p(x)$	$\forall x : q(x)$
$p(X)$	$q(Y)$
$\square$ $\{X/a\}$ with (6)	$\square$ $\{Y/a\}$ with (7)

Try tableau for $G \models F$:
Start with $G \wedge \neg F$.

$$
\begin{array}{c}
\forall v : (p(v) \vee q(v)) \wedge \neg((\forall x : p(x)) \vee (\forall x : q(x))) \quad (1)\\
\forall v : (p(v) \vee q(v)) \quad (2)\\
p(V) \vee q(V) \quad (3)\\
\neg((\forall x : p(x)) \vee (\forall x : q(x))) \quad (4)\\
\neg(\forall x : p(x)) \wedge \neg(\forall x : q(x)) \quad (5)\\
\neg(\forall x : p(x)) \quad (6)\\
\neg(\forall x : q(x)) \quad (7)\\
\neg p(a) \quad (6)\\
\neg q(b) \quad (6)\\
\text{now resolve (3):}
\end{array}
$$

$$
\begin{array}{ccc}
/ & & \backslash \\
p(V) & & q(V)\\
\Box\ \{V/a\} & \cdots\cdots\cdots\to & q(a)\\
 & & |
\end{array}
$$

$$
\begin{array}{c}
\text{(2) nochmal bereitstellen:}\\
p(W) \vee q(W)
\end{array}
$$

$$
\begin{array}{ccc}
/ & & \backslash \\
p(W) & & q(W)\\
p(b) & \leftarrow\cdots\cdots & \Box\ \{W/b\}
\end{array}
$$

This is an example where some ∀-formula is instantiated twice with some variable "standing for everything".
This can be done arbitrarily often, but it is only useful if it leads to "new findings", i.e., the new variable can be used with a substitution that is different from the previous ones.
The open branch describes a (simple, typical) model of $G \wedge \neg F$.

b) $F \models G$ and $G \models F$ hold, thus also $F \equiv G$. The formula is a logical equivalence5B.
Tableau for $F \models G$:
Start with $F \wedge \neg G$.

$$
\begin{array}{c}
\forall x : ((\exists y : p(y)) \rightarrow q(x)) \quad (1)\\
\neg\forall v, w : (p(v) \rightarrow q(w)) \quad (2)\\
\exists v : \neg\forall w : (p(v) \rightarrow q(w))\\
\neg\forall w : (p(a) \rightarrow q(w))\\
\neg(p(a) \rightarrow q(b))\\
p(a)\\
\neg q(b)\\
\text{resolve (1):}\\
(\exists y : p(y)) \rightarrow q(X)
\end{array}
$$

$$
\begin{array}{cc}
/ & \backslash \\
\neg\exists y : p(y) & q(X)\\
\neg p(Y) & \Box\ \{X/b\}\\
\Box\ \{Y/a\} &
\end{array}
$$

Tableau for $G \models F$:
Start with $G \wedge \neg F$.

$$\forall v, w : (p(v) \rightarrow q(w)) \quad (1)$$
$$\neg\forall x : ((\exists y : p(y)) \rightarrow q(x)) \quad (2)$$
$$\neg((\exists y : p(y)) \rightarrow q(a))$$
$$\exists y : p(y)$$
$$\neg q(a))$$
$$p(b)$$
(1) auflösen:
$$p(V) \rightarrow q(W)$$
$$/ \qquad\qquad \backslash$$
$$\neg p(V) \qquad q(W)$$
$$\square\ \{V/b\} \quad \square\ \{W/a\}$$

c) For an intuitive idea, replace $p(a, b)$ by "a knows b", i.e., "every a knows some b" vs. "there is some b that is known by every a".
$G \models F$ holds:
If there is some w_0, such that for each v, $p(v, w)$ holds, then for each x there is some y (namely, w_0), such that $p(x, y)$ holds.
$F \models G$ does not hold:
Let $\mathcal{I} = (I, \mathcal{D})$ with $\mathcal{D} = \{1, 2\}$ and $I(p) = \{(1, 2), (2, 1)\}$.
$\mathcal{I} \models F$, but not $\mathcal{I} \models G$.

Thus, F and G are not equivalent.
Tableau for $F \models G$:

$$F \wedge \neg G$$
$$\forall x \exists y : p(x, y) \quad (1)$$
$$\neg\exists w \forall v : p(v, w) \quad (2)$$
$$\forall w : \neg\forall v : p(v, w) \quad (aus(2))$$
$$\neg\forall v : p(v, W)$$
$$\exists v : \neg p(v, W)$$
$$\neg p(f(W), W)$$
"choose $f(W)$ depending on W such that $\neg p(v, W)$ holds"
now resolve (1):
$$\exists y : p(X, y)$$
$$p(X, g(X))$$

Here, it becomes clear that the tableau cannot be closed due to the non-matching term structure of the arguments.
Tableau for $G \models F$:

$$G \wedge \neg F$$
$$\exists w \forall v : p(v, w) \quad (1)$$
$$\neg\forall x \exists y : p(x, y) \quad (2)$$
$$\exists x : \neg\exists y : p(x, y) \quad (3)$$
$$\neg\exists y : p(a, y) \quad (4)$$
$$\neg p(a, Y) \quad (5)$$
now resolve (1):
$$\forall v : p(v, b)$$
$$p(V, b)$$
$$\square\ \{V \rightarrow a, Y \rightarrow b\}$$

(5) translates to "for a, for all Y $\neg p(a, Y)$ holds" – "a knows nobody". (7) translates to "for each V, $p(V, b)$ holds" – "everybody knows b". Thus, this must also hold for $V = a$, having a contradiction to (5).

Exercise 2 (FOL: Erfüllbarkeit + Tableau) Consider the formula

$$F = \forall x : (p(x) \wedge \exists y(r(x, y) \wedge \neg p(y)))$$

Is it satisfiable or not?

Prove your decision by the tableau calculus

The formula is unsatisfiable.

Tableau for F:

$$\begin{array}{c}
\forall x : (p(x) \wedge \exists y(r(x, y) \wedge \neg p(y))) \quad (1) \\
p(X) \wedge \exists y(r(X, y) \wedge \neg p(y)) \quad (2) \\
p(X) \quad (3) \\
\exists y(r(X, y) \wedge \neg p(y)) \quad (4) \\
r(X, f(X)) \wedge \neg p(f(X)) \quad (5) \\
r(X, f(X)) \quad (6) \\
\neg p(f(X)) \quad (7) \\
\text{can not yet be closed: when replacing } X \text{ by } f(X), \\
\text{there is } p(f(X)) \text{ vs. } \neg p(f(f(X))). \\
\text{Instantiate (1) once more with a new variable:} \\
p(Z) \wedge \exists y(r(Z, y) \wedge \neg p(y)) \quad (2') \\
p(Z) \quad (3') \\
\exists y(r(Z, y) \wedge \neg p(y)) \quad (4') \\
\text{and now close (7) against (3'):} \\
\square \; \{Z \rightarrow f(X)\}
\end{array}$$

This exercise is (another) example where it is necessary to instantiate a universally quantified formula several times to apply it to several "objects" (here an arbitrary (X), and the corresponding $y = f(X)$).

Comparison: do the proof for the equivalent formula

$$F' = (\forall x : p(x)) \wedge \forall x : (\exists y(r(x, y) \wedge \neg p(y))) \; .$$

This shows that it is more efficient to keep the scope of a quantifier small and to use different scopes for separate statements.
