FOL

- Syntax of Terms and Formulas

fct symbols c, c_1, alice, germany, $f/1$, $g/2$

$\hookrightarrow g(c, \text{alice})$

predicates true, false, a, b

$p/1$, $q/2$, $\text{cond}/6$

$\hookrightarrow p(\text{alice})$, $q(t_1, t_2)$

- Semantics

$\mathcal{S} = (I, \mathcal{D})$

$\hookrightarrow$ is interpretation of fct symbols:

$c \mapsto$ domain $\mathcal{D}$

$f/n \mapsto$ functions $\mathcal{D}^n \to \mathcal{D}$

over $\mathcal{D}_B \leftarrow$ | interpretation of predicates:

$p/n \subset \mathcal{D}^n$ or mapping $\mathcal{D}^n \to t/f$

Nov 6-10:09

tautology: formula that holds in every structure

Slide 4/11:

different ways to prove that

$$\underbrace{\left((\forall x: p(x) \to q(x))^{A_1} \wedge (\forall y: q(y) \to r(y))^{A_2}\right)}_{A} \to \underbrace{(\forall z: p(z) \to r(z))}_{B}$$

Nov 6-10:23

1st way: mathematically $A \to B$

Consider an arbitrary structure $\mathcal{S} = (I, \mathcal{D})$

show that $\mathcal{S} \models A \to B$.

Check whether $\mathcal{S} \models \neg A$ or $\mathcal{S} \models B$

we are done if $\mathcal{S} \models \neg A$.

So: consider the rest, namely

$\mathcal{S} \models A$, i.e.

$\mathcal{S} \models A_1 \wedge A_2$

$\Longleftrightarrow$

$\mathcal{S} \models A_1$ and $\mathcal{S} \models A_2$

$\hookrightarrow$

Nov 6-10:26

: $\mathcal{S} \models_\emptyset A_1$ and $\mathcal{S} \models_\emptyset A_2$

$\Longleftrightarrow$ $\mathcal{S} \models_\emptyset \forall x\, (p(x) \to q(x))$

and $\mathcal{S} \models_\emptyset \forall y\, (q(y) \to r(y))$

$\Longleftrightarrow$ for all $d \in \mathcal{D}$: $\mathcal{S} \models_{\emptyset^d_x} p(x) \to q(x)$

and for all $e \in \mathcal{D}$: $\mathcal{S} \models_{\emptyset^e_y} q(y) \to r(y)$

$\Longleftrightarrow$ for all $d \in \mathcal{D}$: $\mathcal{S} \models_{\emptyset^d_x} \neg p(x)$ or $\mathcal{S} \models_{\emptyset^d_x} q(x)$

and for all $e \in \mathcal{D}$: $\mathcal{S} \models_{\emptyset^e_y} \neg q(y)$ or $\mathcal{S} \models_{\emptyset^e_y} r(y)$

$\Longleftrightarrow$ for all $d \in \mathcal{D}$: $(d \notin I(p)$ or $d \in I(q))$

$d = e$

and $(d \notin I(q)$ or $d \in I(r))$

mathematical conclusion

for all $d \in \mathcal{D}$: case a) $d \notin I(p)$

or case b) (relevant if $d \in I(p)$)

$\Rightarrow d \in I(q)$

combine with

~~case b$_1$) $d \notin I(q)$~~ $\lightning$

case b$_2$) $d \in I(r)$

$\Longleftrightarrow$ lift to $\models$-level

for all $d \in \mathcal{D}$: $d \notin I(p)$ or $d \in I(r)$

$\Longleftrightarrow$ for all $d \in \mathcal{D}$. $\mathcal{S} \models_{\emptyset^d_z} \neg p(z)$ or $\mathcal{S} \models_{\emptyset^d_z} r(z)$

$\Longleftrightarrow$ for all $d \in \mathcal{D}$ $\mathcal{S} \models_{\emptyset^d_z} p(z) \to r(z)$

$\Longleftrightarrow$ $\mathcal{S} \models \underbrace{\forall z: p(z) \to r(z)}_{B}$

$\Longleftrightarrow \mathcal{S} \models B$ $\Rightarrow$ every structure that satisfies A also satisfies

$\Rightarrow$ $\mathcal{S} \models \neg A \vee B$

$\Longleftrightarrow$ $\mathcal{S} \models A \to B$

$\square$

Nov 6-10:31

2nd way: shorter mathematical proof

before: forward proof: every $\mathcal{I} \models A \to B$

this one: refutation proof:

there is no $\mathcal{I}$: $\mathcal{I} \not\models A \to B$

assume $\mathcal{I} \not\models A \to B$

i.e. $\mathcal{I} \models A$ but $\mathcal{I} \not\models B$

Nov 6-10:53

$\mathcal{I} \models A$ and $\mathcal{I} \not\models B$

$\mathcal{I} \models \underbrace{(\forall x: p(x) \to q(x))}_{A_1} \wedge \underbrace{(\forall y\, q(y) \to r(y))}_{A_2}$

and $\mathcal{I} \not\models \forall z\; p(z) \to r(z)$

$\Rightarrow$ there is some $d_0 \in \mathcal{D}$: $d_0 \in I(p)$ and $d_0 \notin I(r)$

$\Rightarrow$ from A_1:

for all $d \in \mathcal{D}$: $\mathcal{I} \models_{\alpha_x^d} p(x) \to q(x)$

take d_0 from above: ~~$d \notin I(p)$~~ or $d \in I(q)$

~~$d_0 \in I(p)$~~

$\Rightarrow d_0 \in I(q)$

from A_2 .... conclude that $d_0 \in I(r)$ ↯

$\Rightarrow$ the assumed structure $\mathcal{I}$ cannot exist

Nov 6-10:56

3rd way: pure symbolic reasoning
(later → Tableau proofs)

$\neg(A \to B)$

$A = A_1 \wedge A_2$ assumption

$\cancel{A}$ $A_1 \wedge A_2$

$\cancel{\neg B}$ $\neg(\forall x\,(p(x) \to r(x)))$

A_1

A_2

$\exists x\,(p(x) \wedge \neg r(x))$

some element d of the domain
take a new constant symbol c
and interpret c by $d \in D$

$p(c) \wedge \neg r(c)$

$p(c)$

$\neg r(c)$

$\forall x\,(p(x) \to q(x))$

X^c a new Tableau Variable, can be replaced by any term $X \to c$

$p(X^c) \to q(X^c)$

$\neg p(X^c) \vee q(X^c)$

$\neg p(c) \vee q(c)$

$\neg p(c)$ | $q(c)$

$\square$

$\forall y\,(q(y) \to r(y))$

$q(y) \to r(y)$

replace y by c

$q(c) \to r(c)$

$\neg q(c)$ | $r(c)$

$\square$ $\square$

purely symbolic refutation proof

Nov 6-11:04

Slide 4/12 Ex. 1

F: $\forall x\, p(x) \vee \forall x\, q(x)$

G: $\forall x\,(p(x) \vee q(x))$

$\Downarrow$ $\cancel{\Uparrow}$

Nov 6-11:19

$F: \forall x \; ((\exists y : p(y)) \rightarrow q(x))$

if $I(p) \neq \emptyset \rightarrow I(q) = \mathcal{D}$

$G: \forall v \; \forall w : \; p(v) \rightarrow q(w)$

any $d^{\in \mathcal{D}}$ such that $d \in I(p)$ (*)

then every w s.t. $c \in I(q)$

$\Longleftrightarrow$

if $I(p) \neq \emptyset$

then $I(q) = \mathcal{D}$

$F \Leftrightarrow G$

Nov 6-11:22

Ex 3:

$F: \forall x \; \exists y \; p(x,y)$ "every x is married to some y"

$G: \exists v \; \forall w \; p(v,w)$ "There is some x that is married with every y"

$F \not\Rightarrow G \quad \mathcal{I}_1 = (I_1, \mathcal{D}_1).$

$\mathcal{D}_1 = (c_1, c_2)$

$I(p) = \{ (c_1, c_2), (c_2, c_1) \}$

$\mathcal{I}_1 \models F$

$\mathcal{I}_1 \not\models G$

$G \not\models F : \; \mathcal{I}_2 = (I_2, \mathcal{D}_2)$

$\mathcal{D}_2 = \{ d_1, d_2 \}$

$\mathcal{I}_2 \models G \quad I(p) = \{ (d_1, d_1), (d_1, d_2) \}$

$\mathcal{I}_2 \not\models F$

Nov 6-11:27